Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Sound and Vibration 293 (2006) 426–439
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Determining the elastic constants of isotropic materials by modal vibration testing of rectangular thin plates

Marco Alfano*, Leonardo Pagnotta

Department of Mechanical Engineering, University of Calabria, 87036 Rende, Italy

Received 11 February 2005; received in revised form 12 October 2005; accepted 13 October 2005
Available online 4 January 2006

Abstract

The paper presents a procedure whereby Poisson's ratio and the dynamic Young's modulus of isotropic and homogeneous materials are determined using two of the first four frequencies of natural vibration in thin rectangular plates. The procedure is based on suitable approximate relationships, relating the resonance frequencies to the elastic constants of the material. These relations were derived from those of Warburton by taking into account a correction factor obtained by an extensive series of numerical analyses carried out by a finite element code. In order to verify the procedure, a comparison with reference solutions has been made.

1. Introduction

Determining the elastic properties of materials by analysing their dynamic response is a well-known technique. Its non-destructive and economical nature, the accuracy of the results provided, simplicity, and ease of implementation make it very attractive for both research and industrial contexts.

The applicability of the resonance method depends on knowledge of the frequency equations relating the natural frequencies of a suitable test specimen to the dynamic elastic properties and the density of the material. These equations are solutions of a differential equation, which, generally, depend on the boundary conditions and on the shape of the specimen in a very complicated way. Analytical closed form solutions are, for these reasons, limited to simple geometries and boundary conditions [1,2].

Procedures and recommendations for the elastic characterization of homogeneous and isotropic materials using free-edge test specimens like slender bars (rectangular cross section), rods (cylindrical cross section), and circular plates are specified in ASTM Standards [3,4], while methods used to characterize the elasticity of cylindrical samples and thin square plates are proposed in Refs. [5] and [6], respectively.

Recently, a great number of techniques for the identification of the elastic properties of both isotropic or orthotropic materials have been proposed. In these techniques, the response of a numerical model of the specimen is correlated with the experimental observations of its real structural behaviour. Unknown material parameters in the numerical model are updated until the computed structural behaviour matches the

*Corresponding author. Tel.: +39 0984 494 156; fax: +39 0984 494 673.
E-mail address: alfano@unical.it (M. Alfano).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.10.021

experimental observations as closely as possible. The values of the parameters used in the numerical model in the last computation are the results of the identification procedure and yield the elastic properties of the specimen.

In principle, the approach makes it possible to identify all the elastic properties simultaneously from a single experiment and without damaging the structure. The measured data are generally the first natural frequencies. The Rayleigh–Ritz type [7–12] or Rayleigh type [13–16] analytical approaches, the finite element method (FEM) [17–22] or the approximate method based on the concept of sinusoidal equivalent length [23,24] have been adopted.

It must be pointed out that in almost all the methods mentioned above, initial estimates of the engineering constants must be available to start the procedures for the iteration or the optimization processes necessary for the determination of the elastic properties.

An original method allowing the direct determination of the flexural stiffness from natural frequencies and modal shape measurements of plate specimens of any shape, which does not require initial estimates of the stiffness nor iterative computati ons, is proposed in Refs. [25,26]. This methodology has, unfortunately, the disadvantage of requiring complicated techniques to measure the modal shapes.

However, the elastic constants could be determined more simply, without procedures for the iteration or the optimization processes and this could be done by using any flat plates, if frequency equations were known.

In the present paper the feasibility of using free rectangular thin plates has been investigated. It is well known that no closed-form analytical solutions exist to the eigenvalue problem in this case.

To seek practical solutions, many researchers have resorted to numerical approximations. For example, the Ritz energy method provides accurate solutions. However, it depends largely on the choice of the global admissible functions representing the displacement fields. Well-known existing functions have been expressed in terms of finite series such as trigonometric functions [27–29]. Warburton [28] used characteristic beam vibration functions to obtain, for any boundary conditions, a simple frequency equation relating the natural frequency to the dimensions of the plate, density and elastic constants of the material. These formulas are very useful from an engineering point of view but they are too approximate to be used in a procedure for the determination of the elastic constants. In this paper correction factors for Warburton's formulas are proposed. The correction factors were obtained using the results of an extensive series of numerical dynamic analyses carried out by a commercial finite element code. Therefore, the paper describes a procedure whereby, using the corrected frequency equations, Poisson's ratio v and the dynamic Young's modulus E of free rectangular thin plates are determined. The procedure requires the measurement of two of the first four natural frequencies and, in some cases, the corresponding modal shapes.

2. Approximated frequency expressions

There has been a great deal of research published on the flexural vibrations of rectangular plates. Classical analytical methods have been used to deal with the flexural vibration of thin isotropic plates with different edge conditions. Exact analytical solutions of the governing differential equations have been determined for the case of a rectangular plate that is simply supported at all four edges or having two opposite edges simply supported with any conditions at the other edges [1]. For rectangular plates with other combinations of edge conditions, the solutions are more complicated. To seek practical solutions many researchers have resorted to various approximate analytical methods.

Warburton [28] used characteristic beam vibration functions in Rayleigh's method in order to obtain, for any boundary conditions and for each mode of vibration, a very useful, simple and approximate formula expressing natural frequency in terms of dimensions of the plate, density and elastic constants of the material. Such a formula assumes the following form:

$$f = \frac{\pi}{2}\sqrt{\frac{D}{\rho t}}\frac{\lambda}{a^2}, \tag{1}$$

where $D = Et^3/[12(1 - v^2)]$ is the flexural stiffness of the plate, E and v are the elastic constants, ρ is the density of the material, t is the thickness and f is the natural frequency while λ is a non-dimensional factor

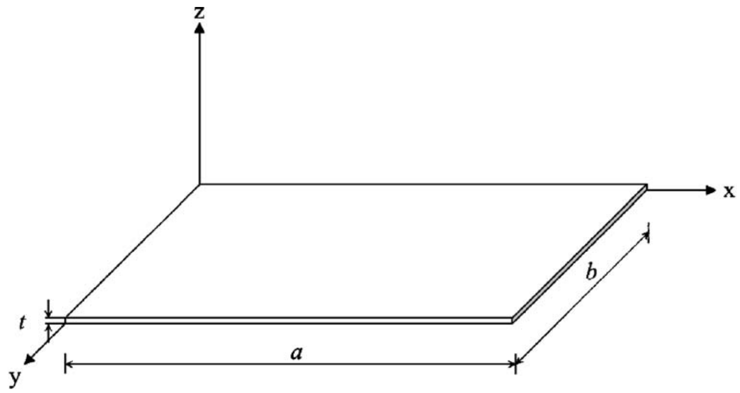

Fig. 1. Schematic representation of the specimen.

obtained using a table reported in Ref. [28]. The frequency factors λ depend on the ratio a/b (a and b are the lengths of the sides of the plate, see Fig. 1), and also upon v, if one or more edges of the plate are free.

Eq. (1) together with the knowledge of the λ expressions allows the straightforward calculation of natural frequencies of plates having any combination of free, clamped, or simply supported edges, if the elastic constants are given. Vice versa, they also allow the elastic constants to be determined if natural frequencies are measured.

The accuracy of the frequencies calculated by Eq. (1) is excellent for plates having no free edges. However, if one or more free edges exist, then the accuracy can decrease significantly. Warburton's formulas are, still today, the only ones that can be found in the published literature which take into account, for each resonant mode of vibration and in a concise way, the effects of both the a/b ratio and Poisson's ratio variations on the resonant frequency.

Leissa [2,29] presented more accurate analytical results for the free flexural vibration of isotropic rectangular plates with $v = 0.3$. In this case the Ritz method with 36 terms containing the products of beam functions was applied.

Today, numerical methods play a very important role in dealing with complicated structural dynamic problems. In particular, the FEM is extremely well suited to the computer solution of free and forced vibration problems associated with complex plate or shell structures and for isotropic materials it gives very accurate results. In the present paper, FEM was used to determine the dependence of the frequency factors λ upon v in order to improve the accuracy of the Warburton formulas so that they are suitable in a procedure determining the elastic properties.

The case of thin rectangular plate with free edges was explored for various modes of flexural vibration and for different a/b ratios. The results were compared with those obtained by Warburton and when possible with those of Leissa.

3. Finite element models and modal analysis

The values of the frequency factors λ upon v were derived from an extensive series of numerical analyses carried out by a commercial finite element code. In particular, the values of the resonant frequencies were calculated for various values of v and for a defined geometry of the plate, and for fixed values of material density and Young's modulus. Then, λ was computed, for each v, by mathematically inverting Eq. (1). A proper numerical investigation have confirmed that, as reported in Ref. [28], λ does not depend on E. In addition, it has been verified that for a thin plate ($a/t \geqslant 100$) λ does not depend on the ratio a/t.

The normal mode analysis for predicting the natural frequencies of the plates was carried out, neglecting damping effects, by using Solution 103 of the general-purpose finite element code MSC/NASTRAN, with MSC/PATRAN as the pre- and post-processor. The Lanczos extraction method [30] was adopted.

2-D finite element models of four free thin plates with aspect ratio (a/b) equal to 1, 1.5, 2.0, 2.5, respectively, have been developed. Quadratic eight node (CQUAD8) elements were used and both the effect of the transverse shear deformation and of the rotary inertia were neglected. The boundary conditions for the models were all edge free. A convergence study was done to determine the mesh density at which the values of the first 10 fundamental frequencies converge. The number of elements assumed for the four plates examined were 40×40, 40×60, 40×80 and 40×120, respectively. In order to verify the accuracy of the models, they were used to determine the undamped free flexural vibrations of aluminium plates simply supported on all edges. Eigenfrequencies given by the FEM code were compared with those obtained by the following exact analytical solutions [1]:

$$f_{(m,n)} = \frac{\pi}{2}\sqrt{\frac{D}{\rho t}}\left[\left(\frac{m-1}{a}\right)^2 + \left(\frac{n-1}{b}\right)^2\right], \quad (2)$$

where m and n are the numbers of nodal lines along directions x and y, respectively. The results have shown that the difference between frequencies given by FEM and Eq. (2) is less than 0.05%.

Denoting the mode of vibration by the numbers of nodal lines is suitable where the nodal lines are approximately parallel to the sides of the rectangle. This is the case in rectangular plate with free edges, but it is well known that non-parallel patterns can be observed for a free-edged square plate. In this case, in fact, if $m = n$ or $m - n = \pm 1, \pm 3, \pm 5, \ldots$ the normal modes of vibration are of the type (m, n) with nodal lines approximately parallel to the sides, while, when $m - n = \pm 2, \pm 4, \pm 6, \ldots$, the normal modes are of the type $((m, n) \pm (m, n))$, with patterns that do not consist of lines parallel to the sides of the plate.

3.1. Free square plates

The first four modes of vibration of a square isotropic steel thin plate ($\nu = 0.3$) with free edges are illustrated in Fig. 2 along with the modal designations and the relative frequencies (with respect to the first frequency).

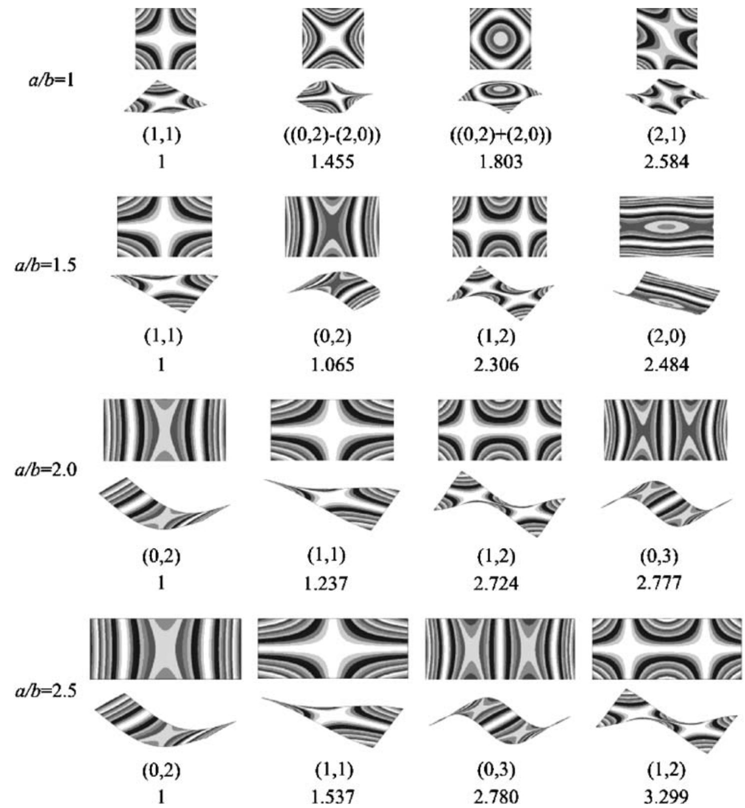

Fig. 2. First four modal shapes and relative frequencies (with respect to the fundamental frequency) for a free steel plate ($\nu = 0.3$).

All the vibration modes reported in the figure were obtained by the finite element code. The (1,1) mode ('+' mode), the lowest mode, is pure twisting motion; the frequency of the ((2,0) + (0,2)) mode is greater than the ((2,0)−(0,2)) mode because of the Poisson coupling, in particular these modes have frequencies which differ by

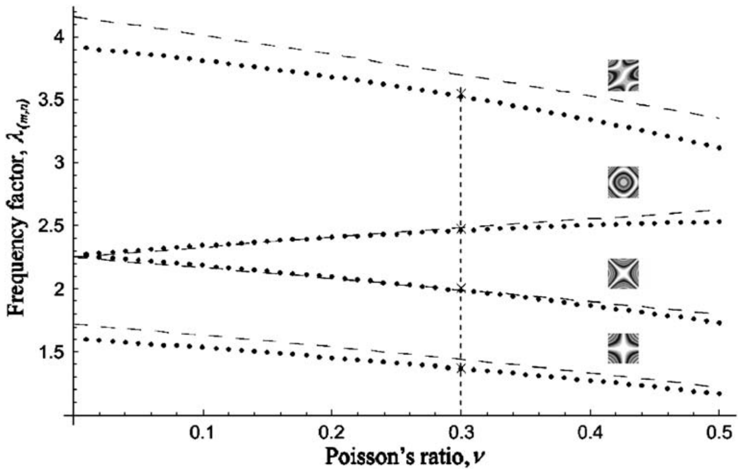

Fig. 3. Variation in the frequency factors with ν for $a/b = 1.0$; (––) Warburton [28], (×) Leissa [29], (●) FEM.

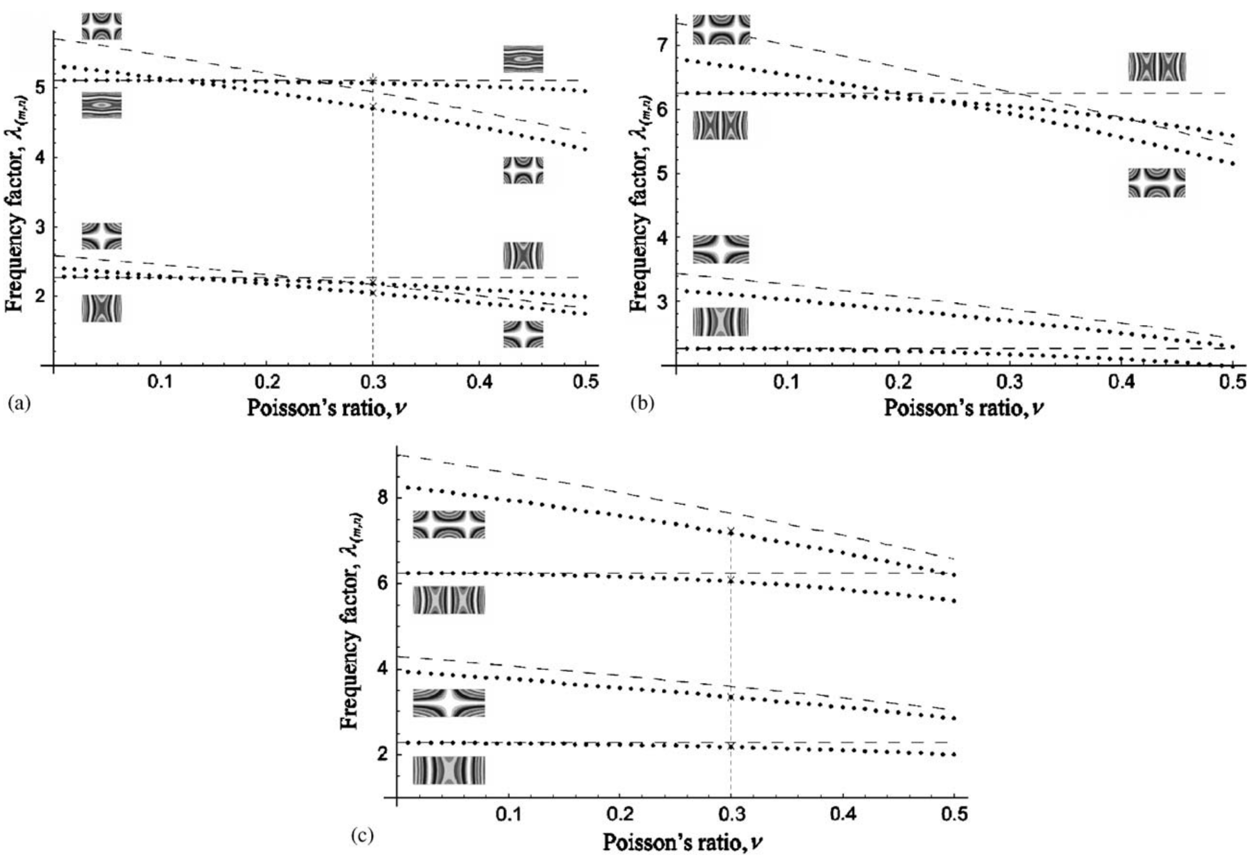

Fig. 4. Variation in the frequency factor with ν: (a) $a/b = 1.5$, (b) $a/b = 2.0$, (c) $a/b = 2.5$; (––) Warburton [28], (×) Leissa [29], (●) FEM.

an amount which is a measure of the strength of Poisson's ratio coupling. They are generally referred to as the '*X*' mode and the '*O ring*' mode. The modes (2,1) and (1,2) may be similarly replaced by the non-degenerate combination modes ((2,1) ± (1,2)), but all three modes have the same frequency, because the Poisson coupling does not favour one or the other combination.

The variation in the lowest four frequency factors λ upon v is reported in Fig. 3. In particular, the graph compares the variation of the frequency factors calculated by FEM with the Warburton frequency factors and, the values obtained by Leissa for $v = 0.3$. The FEM values were obtained by means of the procedure described in the previous section, varying v between 0.01 and 0.5 in steps of 0.01. These values are reported in the appendix in the first four columns of Table A1. It can be observed that only the numerically calculated factors corresponding to the ((2,0) ± (0,2)) modes of vibration are in good agreement with the Warburton values, while, an excellent agreement can be found with the Leissa values.

3.2. Free rectangular plates

The first four vibration modes of a free-edged, rectangular isotropic steel plate ($v = 0.3$), for three different values of a/b (1.5, 2.0, and 2.5) are illustrated in Fig. 2. Note that the sequence and precise details of mode shapes depend on the particular values of elastic constants and geometrical dimensions. Fig. 4 shows the variations in the first four frequency factors with Poisson's ratio for each a/b examined. In the same figure FEM results are compared with those obtained by Warburton and Leissa.

For $a/b = 1.5$ (Fig. 4a), it can be observed that the modal shape of the lower natural frequency is (0,2) for v less than approximately 0.13 while for larger v the modal shape is (1,1). Vice versa, the second resonant frequency vibrates the plate in the mode (1,1) for v less than approximately 0.13 while for larger v the modal shape becomes (0,2). The modal shapes (2,0) and (1,2) are associated to the third or to the fourth natural frequency according to the value of v (modes inversion occurs at $v \cong 0.12$)·

For $a/b = 2$ (Fig. 4b), only the reversing between the two modes (1,2) and (0,3) can be observed (inversion occurring at $v \cong 0.24$). No modal reversals occur for a/b as large or larger than 2.5 (Fig. 4c). In these cases, as for the case of square plate, the modal shape associated to any order of frequency does not change varying v.

4. Procedures for the elastic characterization

This section describes a procedure for the identification of elastic properties of isotropic materials from the measurement of the frequencies of two of the first resonant modes of free thin rectangular plates.

In principle, Poisson's ratio can be determined by the frequency ratio $f_{(h,k)}/f_{(l,j)}$ of two generic (h, k) and (l, j) resonant modes. In fact, as can be deduced from Eq. (1), the frequency ratio $f_{(h,k)}/f_{(l,j)}$ is equal to the frequency factors ratio $\lambda_{(h,k)}/\lambda_{(l,j)}$ and they are only dependant on v for a given value of a/b. Then, v can be straightforwardly determined from $f_{(h,k)}/f_{(l,j)}$ if this dependence is known.

The value of v could be used to evaluate Young's modulus. In fact, inverting Eq. (1) and introducing the mass m of the specimen, Young's modulus is expressed as

$$E = \frac{48}{\pi^2}\left(\frac{f}{\lambda(v)}\right)^2 \frac{ma^3(1 - v^2)}{bt^3}, \tag{3}$$

where f is one of the two resonant frequencies considered.

The procedure for determining v and E can be summarized in three steps. First, two resonance frequencies are experimentally measured, their modal shape identified and the frequency ratio is computed. Second, Poisson's ratio is found by a graph or by a proper numerical table from the frequency ratio computed in the previous step. Finally, E is calculated by using Eq. (3).

In any case, the accuracy of the results depends on the accuracy of the natural frequencies measurement. The first low resonant frequencies are always preferable, because the measurement of the highest natural frequencies, generally, is more difficult and not very accurate. However, the accuracy of the results also depends on the sensitivity of v to the changes in the chosen frequency factor ratio (i.e., the slope of the curve relating Poisson's ratio with the frequency factor ratio). As it will be seen in the following paragraphs, ratios related to large slope variations should be avoided.

In order to verify the present procedure, experimental data given in the bibliography were considered; in such a way a comparison with reference solutions has been made.

4.1. Application to a square plate

The variations of v with frequency factor ratios, for all the possible combinations of the first four modes of vibration of a square plate, are illustrated in Fig. 5. In principle, the value of v could be graphically obtained from the value of any $f_{(h,k)}/f_{(l,j)}$. It is sufficient to find the intersection point of the $\lambda_{(h,k)}/\lambda_{(l,j)}$ vertical line and the curve. The ordinate of this point is the value of v for the material. If higher precision is desired, tabulated values reported in Table A1 should be used.

Practically, the sensitivity of v to the changes in frequency factor ratio, represented from the slope of the curve considered, should be taken into account. Relatively flatter curves propagate the experimental errors on frequency measurements less and are preferable. The sensitivity plays a very important role in choosing the more suitable ratio $\lambda_{(h,k)}/\lambda_{(l,j)}$ to compute v. The slope variations of the curves with the frequency factor ratio show $f_{((2,0)+(0,2))}/f_{(1,1)}$ and $f_{(2,1)}/f_{((2,0)+(0,2))}$ as the more suitable frequency ratios to use for the calculation of v because they are less prone to experimental errors.

From the same figure we can observe that $f_{((2,0)+(0,2))}/f_{((2,0)-(0,2))}$ is also appropriate. A good agreement was found between this function and the well-known Warburton approximate function [28]

$$v \cong 1.389 \frac{f^2_{((2,0)+(0,2))}\Big/f^2_{((2,0)-(0,2))} - 1}{f^2_{((2,0)+(0,2))}\Big/f^2_{((2,0)-(0,2))} + 1}. \tag{4}$$

By applying the theory of the uncertainty propagation it can be shown that, when one of the three aforementioned ratios is used for characterizing a material with v around 0.3, a relative error of 0.1% on the measured frequencies implies an error on the calculated elastic constants of less than about 1%. The estimated errors are much higher when the other frequency ratios are used.

To show how the procedure works we considered the square aluminium plate investigated in the paper of Deobald and Gibson [10]. The dimensions of the plate and the density of the material are reported in Table 1. In the same table in increasing order the first four natural frequencies measured are also reported. The modal shapes corresponding to these frequencies are directly identified from Fig. 3 ($f_{\rm I} = f_{(1,1)}$, $f_{\rm II} = f_{((2,0)-(0,2))}$, $f_{\rm III} = f_{((2,0)+(0,2))}$ and $f_{\rm IV} = f_{(2,1)}$). The values of v determined by using Table A1 from the three more suitable

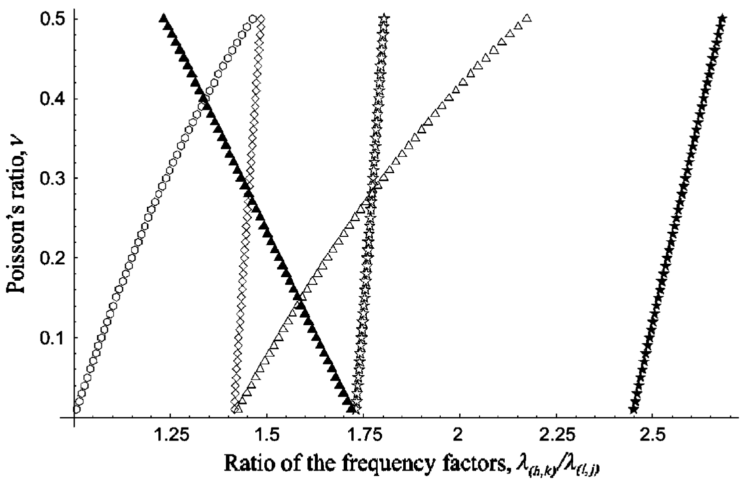

Fig. 5. Variation in v with the ratio of the frequency factors for $a/b = 1.0$; (◇) $\lambda_{((0,2)-(2,0))}/\lambda_{(1,1)}$, (△) $\lambda_{((0,2)+(2,0))}/\lambda_{(1,1)}$, (⬡) $\lambda_{((0,2)+(2,0))}/\lambda_{((0,2)-(2,0))}$, (★) $\lambda_{(2,1)}/\lambda_{(1,1)}$, (☆) $\lambda_{(2,1)}/\lambda_{((0,2)-(2,0))}$, (▲) $\lambda_{(2,1)}/\lambda_{((0,2)+(2,0))}$.

Table 1
Dimensions, density and natural frequencies of the aluminium plate investigated in Ref. [10]

Length, a (cm)	Width, b (cm)	Thickness, t (cm)	Density, ρ (g/cm^3)	a/b	f_{I} (Hz)	f_{II} (Hz)	f_{III} (Hz)	f_{IV} (Hz)
25.4	25.4	0.316	2.77	1	156.70	232.50	300.40	411.70

Table 2
Values of the elastic constants obtained with the present method for the aluminium plate investigated in Ref. [10]

Frequency ratio			v	Young modulus, E (GPa)	
$f_{((0,2)+(2,0))}/f_{(1,1)}$	$f_{\mathrm{III}}/f_{\mathrm{I}}$	1.917	0.369	$f_{((0,2)+(2,0))}$	70.61
				$f_{(1,1)}$	69.71
$f_{((0,2)+(2,0))}/f_{((0,2)-(2,0))}$	$f_{\mathrm{III}}/f_{\mathrm{II}}$	1.292	0.354	$f_{((0,2)+(2,0))}$	71.86
				$f_{((0,2)-(2,0))}$	71.80
$f_{(1,2)}/f_{((0,2)+(2,0))}$	$f_{\mathrm{IV}}/f_{\mathrm{III}}$	1.371	0.366	$f_{(1,2)}$	71.05
				$f_{((0,2)+(2,0))}$	70.86
Mean values			0.363	—	70.98

Table 3
Comparison between the values of the elastic constants calculated with the present method with those obtained in Refs. [10,17]

	[10][a]	[17][a]	Present
E_1 (GPa)	69.50	72.20	70.98
E_2 (GPa)	69.90	73.30	
v	0.361	0.356	0.363

[a]Average values.

ratios $f_{((2,0)+(0,2))}/f_{(1,1)}$, $f_{((2,0)+(0,2))}/f_{((2,0)-(0,2))}$, $f_{(1,2)}/f_{((2,0)+(0,2))}$ are presented in Table 2. The values of v calculated by the remaining frequency ratios $f_{((2,0)-(0,2))}/f_{(1,1)}$, $f_{(1,2)}/f_{(1,1)}$ and $f_{(1,2)}/f_{((2,0)-(0,2))}$ are not reported because they are not accurate enough. Table 2 also illustrates the values of Young's modulus calculated by Eq. (3) for each v and natural frequency. In Table 3 the present results and those obtained from Deobald and Gibson [10], with a modal analysis/Rayleigh–Ritz techniques and using the first five natural frequencies, are compared. A comparison is also made with the results obtained from Hwang and Chang [17] by using the first six natural frequencies with a method combining finite element analysis and optimum design. Both the methodologies are suitable for isotropic and orthotropic materials. The authors treated the isotropic plate as a transversely isotropic material and tried to obtain four elastic constants: the longitudinal Young's modulus E_1, transverse Young's modulus E_2, major Poisson's ratio v_{12} and the in-plane shear modulus G_{12}. It was shown that the four elastic constants approximately satisfy the condition for isotropic materials: $E_1 = E_2$, $G_{12} = E_1/2(1 + v_{12})$.

A good agreement can be observed between the present values and those reported in Refs. [10,17]. The discrepancies between the results and the typical values for aluminium could be essentially due to the fact that natural frequencies reported in Ref. [10] were measured with experimental errors within 1%, and, to a lesser extent, to the fact that the plate examined does not satisfy the condition $a/t = 100$ as required by all the methodologies mentioned above.

4.2. Application to a rectangular plate

The above procedure also works for the elastic characterization of rectangular plates. To prove this assertion, consider the experimental data reported in Ref. [18]. Once again, the specimen analysed is an aluminium plate. It is of rectangular shape with $a/b = 1.456$. The plate dimensions, the density of the material and the natural frequencies measured are reported in Table 4.

Because this plate is not included in the cases previously presented, a proper numerical calibration has been carried out. A comparison between these results and those obtained for $a/b = 1.5$ are illustrated in Fig. 6. The first step of the procedure for the elastic characterization requires the measurement of the frequencies corresponding to two identified modal shapes. Unfortunately, the modal shapes corresponding to the natural frequencies given in Ref. [18] and reported in Table 4 are not specified. Neither does the observation of the graph in Fig. 6 give this information unequivocally (e.g., the third natural frequency f_{III} could correspond to the (2,0) mode of vibration if v is less than 0.12 or to (1,2) mode in the other case). So, the procedure would not seem to be directly applicable. But, if we have additional knowledge such as that the tested material has v greater than 0.12, then Fig. 6 can give a satisfactory solution to the problem of the modal shapes, in fact they will be $f_I = f_{(1,1)}$, $f_{II} = f_{(0,2)}$, $f_{III} = f_{(1,2)}$, $f_{IV} = f_{(2,0)}$. In Fig. 7, the three curves less sensitive to frequency factor ratio error are shown, namely, v versus $\lambda_{(2,0)}/\lambda_{(1,1)}$, $\lambda_{(1,2)}/\lambda_{(0,2)}$ and $\lambda_{(2,0)}/\lambda_{(0,2)}$. The numerical values of the plotted data can be found in Table A2. If a material with v around 0.3 is characterized using one of the three aforementioned ratios, a relative error of 0.1% on the measured frequencies provides errors on the calculated elastic constants of less than 3%. In Fig. 7 the curves relative to the case with $a/b = 1.5$ are also illustrated. It can be seen that, even if the two ratios are almost equal, a significant error can occur in determining v if these last curves are used in place of the former ones.

Now we can apply the procedure to determine Poisson's ratio. The values of v obtained by Table A2 using the three recommended frequency ratios $f_{(2,0)}/f_{(0,2)} (= f_{IV}/f_{II})$, $f_{(1,2)}/f_{(0,2)} (= f_{III}/f_{II})$ and $f_{(2,0)}/f_{(1,1)} (= f_{IV}/f_I)$ are reported in Table 5. In the same table the values of Young modulus calculated from Eq. (3) for each v and for each natural frequency used are also reported. Finally, the comparison between the present results and those obtained by other researchers with different iterative procedures [18–21] are reported in Table 6. The agreement among the sets of results is quite good for Young's modulus but not very good for Poisson's ratio. The discrepancies between the present values of v and those obtained in Refs. [18–21] could be attributed to

Table 4
Dimensions, density and natural frequencies of the aluminium plate investigated in Ref. [18]

Length, a (cm)	Width, b (cm)	Thickness, t (cm)	Density, ρ (g/cm^3)	a/b	f_I (Hz)	f_{II} (Hz)	f_{III} (Hz)	f_{IV} (Hz)
28.1	19.3	0.194	2.688	1.456	112.60	127.90	267.90	286.60

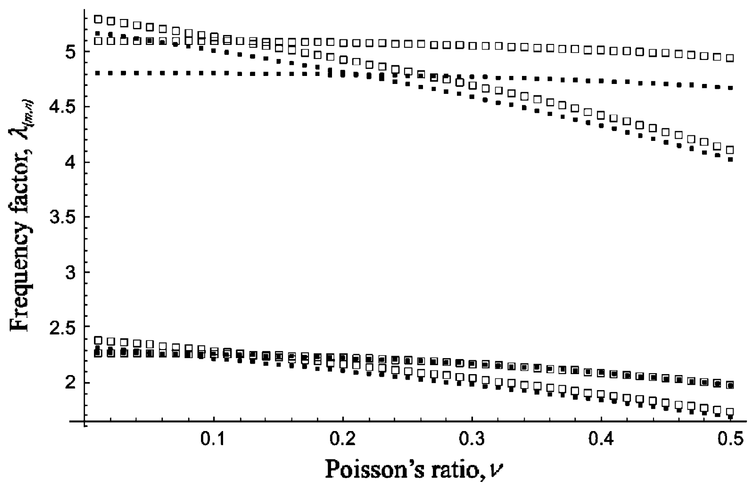

Fig. 6. Variation in the frequency factor with v; (■) $a/b = 1.456$, (□) $a/b = 1.5$.

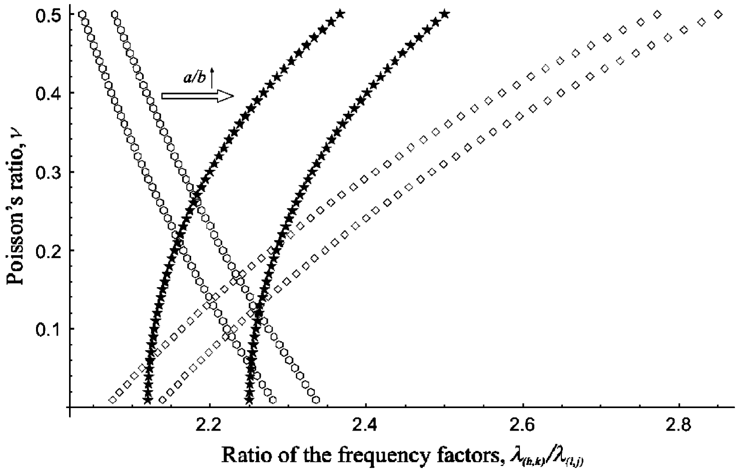

Fig. 7. Variation in ν with the ratio of the frequency factors for the aspect ratios $a/b = 1.456$ and $a/b = 1.5$; (⬡) $\lambda_{(1,2))}/\lambda_{(0,2)}$, (◇) $\lambda_{(2,0)}/\lambda_{(1,1)}$, (★) $\lambda_{(2,0)}/\lambda_{(0,2)}$.

Table 5
Values of the elastic constants obtained with the present method for the aluminium plate investigated in Ref. [18]

Frequency ratio			ν	Young modulus, E (GPa)	
$f_{(2,0)}/f_{(0,2)}$	f_{IV}/f_{II}	2.241	0.365	$f_{(2,0)}$	68.42
				$f_{(0,2)}$	68.45
$f_{(1,2)}/f_{(0,2)}$	f_{III}/f_{II}	2.095	0.347	$f_{(1,2)}$	68.40
				$f_{(0,2)}$	68.48
$f_{(2,0)}/f_{(1,1)}$	f_{IV}/f_{I}	2.545	0.384	$f_{(2,0)}$	67.56
				$f_{(1,1)}$	67.72
Mean values			0.365	—	68.17

Table 6
Comparison between the values of the elastic constants calculated with the present method with those reported in Refs. [18–21]

	[18]	[19]	[20]	[21]	Present
E_1 (GPa)	68.70	69.50	71.30	67.50	68.17
E_2 (GPa)	68.10	67.80	68.80	67.50	
ν	0.340	0.340	0.320	0.356	0.365

the fact that the latter were obtained averaging a larger number of natural frequencies (the first nine natural frequencies).

5. Remarks

In summary, in the case of square plate the order of magnitude of the natural frequencies identifies the shapes of vibration unequivocally. Then, in this case, only the measurement of the resonant frequencies is required.

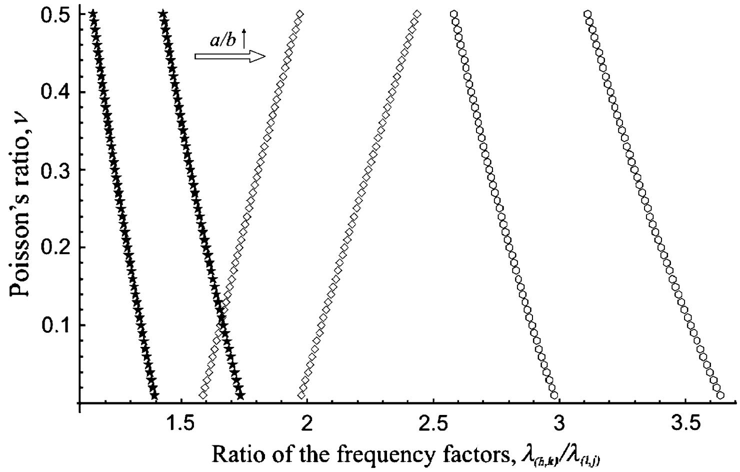

Fig. 8. Variation in v with the ratio of the frequency factors for the aspect ratio $a/b = 2.0$ and $a/b = 2.5$; (◯) $\lambda_{(1,2)}/\lambda_{(0,2)}$, (◇) $\lambda_{(0,3)}/\lambda_{(1,1)}$, (★) $\lambda_{(1,1)}/\lambda_{(0,2)}$.

When rectangular plates with values of the aspect ratio a/b of approximately 1.5 have to be characterized, the procedure can be applied providing it is possible to associate the corresponding modal shapes to each natural frequency measured. In the case that the available experimental technique is only able to measure the resonant frequencies without providing information on modal shapes, the order of magnitude of the frequencies can give the desired information, unambiguously, only when the investigator knows if the value of v is larger or smaller than the value at which the inversion between the modal shape takes place.

In the graph of Fig. 8 the more suitable curves for $a/b = 2.0$ and 2.5 are shown. If, for v around 0.3, one of these curves is used, a relative error of 0.1% on the measured frequencies implies errors on the elastic constants of less than about 3%. When rectangular plates with $a/b = 2.0$ must be tested, the modal shape can be associated to the frequency by using Fig. 4b. However, for this it is necessary to know if v is larger or smaller than 0.23. In that case the ratios $f_{(1,2)}/f_{(0,2)}$ or $f_{(0,3)}/f_{(1,1)}$ can be used to obtain the desired result from the graphs reported in Fig. 8. Ambiguity never arises if the ratio $f_{(1,1)}/f_{(0,2)}$ is used because it is always equal to the ratio between the second and first natural frequencies f_{II}/f_{I} in order of magnitude.

Finally, note that the procedure works without any complications in the cases of rectangular plates with $a/b = 2.5$. In these cases, in fact, the modal shapes corresponding to the natural frequencies can be directly deducible from Fig. 4c.

6. Conclusion

In the paper a procedure is presented that allows the well-known resonance method, for the elastic characterization of homogeneous isotropic materials, to be extended to rectangular thin plate. The procedure requires the measurement of two of the first four natural frequencies and sometimes the determination of the corresponding modal shapes. Poisson's ratio can be determined by using suitable graphs or numerical tables reported in the paper and Young's modulus can be calculated by a proper formula. Five a/b ratios have been examined in the paper, but a numerical methodology is proposed which enables the study of rectangular plates with any other a/b ratio. In principle, the numerical methodology could also be adapted to characterize plates of other shape.

The procedure is suitable for being computerized and, it could be easily integrated in any of the many existing commercial systems for measuring elastic constants by resonant method. Future research could investigate the possibilities of application to thick plate.

Appendix A

The non-dimensional frequency factors (λ) are given in Tables A1 and A2.

Table A1
Non-dimensional frequency factors $\lambda = (2fa^2/\pi)/\sqrt{D/\rho t}$ and their most sensitive ratios for the first four modes of vibration of a square plate

v	$\lambda_{(1,1)}$	$\lambda_{((0,2)-(2,0))}$	$\lambda_{((0,2)+(2,0))}$	$\lambda_{(2,1)}$	$\lambda_{((0,2)+(2,0))}/\lambda_{(1,1)}$	$\lambda_{((0,2)+(2,0))}/\lambda_{((0,2)-(2,0))}$	$\lambda_{(2,1)}/\lambda_{((0,2)+(2,0))}$
0.01	1.5955	2.2590	2.2747	3.9090	1.4257	1.0069	1.7185
0.02	1.5884	2.2511	2.2823	3.8988	1.4369	1.0139	1.7083
0.03	1.5813	2.2430	2.2899	3.8884	1.4481	1.0209	1.6981
0.04	1.5741	2.2349	2.2974	3.8779	1.4595	1.0280	1.6880
0.05	1.5669	2.2266	2.3048	3.8671	1.4709	1.0351	1.6778
0.06	1.5596	2.2183	2.3122	3.8562	1.4826	1.0423	1.6678
0.07	1.5522	2.2098	2.3194	3.8450	1.4943	1.0496	1.6578
0.08	1.5448	2.2013	2.3265	3.8337	1.5060	1.0569	1.6478
0.09	1.5373	2.1927	2.3335	3.8221	1.5179	1.0642	1.6379
0.10	1.5297	2.1839	2.3405	3.8103	1.5300	1.0717	1.6280
0.11	1.5221	2.1751	2.3473	3.7983	1.5421	1.0792	1.6182
0.12	1.5144	2.1662	2.3541	3.7861	1.5545	1.0867	1.6083
0.13	1.5067	2.1571	2.3608	3.7736	1.5669	1.0944	1.5984
0.14	1.4989	2.1480	2.3673	3.7610	1.5794	1.1021	1.5887
0.15	1.4910	2.1387	2.3738	3.7481	1.5921	1.1099	1.5789
0.16	1.4830	2.1293	2.3802	3.7349	1.6050	1.1178	1.5692
0.17	1.4750	2.1199	2.3865	3.7216	1.6180	1.1258	1.5594
0.18	1.4669	2.1103	2.3927	3.7080	1.6311	1.1338	1.5497
0.19	1.4588	2.1005	2.3988	3.6941	1.6444	1.1420	1.5400
0.20	1.4505	2.0907	2.4048	3.6800	1.6579	1.1502	1.5303
0.21	1.4422	2.0808	2.4107	3.6657	1.6715	1.1585	1.5206
0.22	1.4338	2.0707	2.4165	3.6511	1.6854	1.1670	1.5109
0.23	1.4254	2.0605	2.4222	3.6362	1.6993	1.1755	1.5012
0.24	1.4168	2.0502	2.4277	3.6211	1.7135	1.1841	1.4916
0.25	1.4082	2.0397	2.4332	3.6057	1.7279	1.1929	1.4819
0.26	1.3995	2.0291	2.4386	3.5900	1.7425	1.2018	1.4722
0.27	1.3908	2.0184	2.4439	3.5741	1.7572	1.2108	1.4625
0.28	1.3819	2.0076	2.4491	3.5579	1.7723	1.2199	1.4527
0.29	1.3730	1.9966	2.4541	3.5414	1.7874	1.2291	1.4431
0.30	1.3640	1.9855	2.4590	3.5246	1.8028	1.2385	1.4333
0.31	1.3549	1.9742	2.4639	3.5075	1.8185	1.2480	1.4236
0.32	1.3457	1.9628	2.4686	3.4901	1.8344	1.2577	1.4138
0.33	1.3365	1.9513	2.4731	3.4724	1.8504	1.2674	1.4041
0.34	1.3271	1.9396	2.4776	3.4544	1.8669	1.2774	1.3943
0.35	1.3176	1.9277	2.4819	3.4360	1.8837	1.2875	1.3844
0.36	1.3081	1.9157	2.4861	3.4174	1.9005	1.2978	1.3746
0.37	1.2985	1.9035	2.4902	3.3984	1.9178	1.3082	1.3647
0.38	1.2887	1.8912	2.4941	3.3791	1.9354	1.3188	1.3548
0.39	1.2789	1.8787	2.4979	3.3594	1.9532	1.3296	1.3449
0.40	1.2690	1.8660	2.5015	3.3394	1.9712	1.3406	1.3350
0.41	1.2590	1.8531	2.5050	3.3190	1.9897	1.3518	1.3250
0.42	1.2488	1.8401	2.5083	3.2983	2.0086	1.3631	1.3150
0.43	1.2386	1.8269	2.5114	3.2771	2.0276	1.3747	1.3049
0.44	1.2282	1.8135	2.5144	3.2556	2.0472	1.3865	1.2948
0.45	1.2178	1.7999	2.5172	3.2337	2.0670	1.3985	1.2846
0.46	1.2072	1.7861	2.5198	3.2114	2.0873	1.4108	1.2745
0.47	1.1965	1.7721	2.5222	3.1887	2.1080	1.4233	1.2643
0.48	1.1857	1.7580	2.5244	3.1656	2.1290	1.4359	1.2540
0.49	1.1747	1.7436	2.5264	3.1421	2.1507	1.4490	1.2437
0.50	1.1637	1.7289	2.5281	3.1181	2.1725	1.4623	1.2334

Table A2

Non-dimensional frequency factors $\lambda = (2fa^2/\pi)/\sqrt{D/\rho t}$ and their most sensitive ratios for the first four modes of vibration of a rectangular plate ($a/b = 1.456$)

ν	$\lambda_{(0,2)}$	$\lambda_{(1,1)}$	$\lambda_{(2,0)}$	$\lambda_{(1,2)}$	$\lambda_{(2,0)}/\lambda_{(0,2)}$	$\lambda_{(1,2)}/\lambda_{(0,2)}$	$\lambda_{(2,0)}/\lambda_{(1,1)}$
0.01	2.2668	2.3161	4.8053	5.1687	2.1199	2.2802	2.0747
0.02	2.2665	2.3057	4.8052	5.1524	2.1201	2.2733	2.0841
0.03	2.2659	2.2952	4.8051	5.1357	2.1206	2.2665	2.0935
0.04	2.2652	2.2847	4.8048	5.1189	2.1211	2.2598	2.1030
0.05	2.2643	2.2740	4.8045	5.1018	2.1218	2.2531	2.1128
0.06	2.2633	2.2631	4.8041	5.0845	2.1226	2.2465	2.1228
0.07	2.2617	2.2525	4.8037	5.0670	2.1239	2.2404	2.1326
0.08	2.2601	2.2416	4.8031	5.0492	2.1252	2.2341	2.1427
0.09	2.2583	2.2306	4.8025	5.0311	2.1266	2.2278	2.1530
0.1	2.2563	2.2195	4.8018	5.0128	2.1282	2.2217	2.1635
0.11	2.2540	2.2084	4.8011	4.9943	2.1300	2.2157	2.1740
0.12	2.2516	2.1971	4.8002	4.9755	2.1319	2.2098	2.1848
0.13	2.2489	2.1858	4.7993	4.9564	2.1341	2.2039	2.1957
0.14	2.2460	2.1743	4.7983	4.9371	2.1364	2.1982	2.2068
0.15	2.2429	2.1628	4.7972	4.9175	2.1388	2.1925	2.2181
0.16	2.2395	2.1511	4.7960	4.8977	2.1415	2.1870	2.2296
0.17	2.2360	2.1394	4.7947	4.8775	2.1443	2.1814	2.2411
0.18	2.2322	2.1276	4.7933	4.8571	2.1473	2.1759	2.2529
0.19	2.2281	2.1156	4.7919	4.8364	2.1507	2.1706	2.2650
0.2	2.2239	2.1036	4.7903	4.8155	2.1540	2.1653	2.2772
0.21	2.2194	2.0914	4.7942	4.7887	2.1601	2.1577	2.2923
0.22	2.2146	2.0792	4.7869	4.7726	2.1615	2.1551	2.3023
0.23	2.2097	2.0669	4.7850	4.7507	2.1655	2.1499	2.3151
0.24	2.2045	2.0544	4.7831	4.7285	2.1697	2.1449	2.3282
0.25	2.1990	2.0418	4.7810	4.7060	2.1742	2.1401	2.3416
0.26	2.1933	2.0292	4.7788	4.6832	2.1788	2.1352	2.3550
0.27	2.1874	2.0164	4.7764	4.6600	2.1836	2.1304	2.3688
0.28	2.1812	2.0035	4.7740	4.6366	2.1887	2.1257	2.3828
0.29	2.1748	1.9904	4.7714	4.6128	2.1939	2.1210	2.3972
0.3	2.1681	1.9773	4.7687	4.5886	2.1995	2.1164	2.4117
0.31	2.1611	1.9641	4.7658	4.5641	2.2053	2.1119	2.4265
0.32	2.1539	1.9507	4.7628	4.5392	2.2112	2.1074	2.4416
0.33	2.1464	1.9372	4.7596	4.5140	2.2175	2.1031	2.4569
0.34	2.1387	1.9235	4.7563	4.4884	2.2239	2.0987	2.4727
0.35	2.1307	1.9098	4.7528	4.4625	2.2306	2.0944	2.4886
0.36	2.1224	1.8959	4.7492	4.4361	2.2377	2.0901	2.5050
0.37	2.1138	1.8819	4.7453	4.4094	2.2449	2.0860	2.5215
0.38	2.1050	1.8677	4.7413	4.3823	2.2524	2.0819	2.5386
0.39	2.0959	1.8534	4.7370	4.3547	2.2601	2.0777	2.5558
0.4	2.0864	1.8389	4.7326	4.3267	2.2683	2.0738	2.5736
0.41	2.0767	1.8244	4.7279	4.2983	2.2766	2.0698	2.5915
0.42	2.0667	1.8096	4.7230	4.2695	2.2853	2.0659	2.6100
0.43	2.0564	1.7947	4.7178	4.2403	2.2942	2.0620	2.6287
0.44	2.0457	1.7797	4.7124	4.2105	2.3036	2.0582	2.6479
0.45	2.0348	1.7644	4.7067	4.1803	2.3131	2.0544	2.6676
0.46	2.0235	1.7491	4.7007	4.1497	2.3231	2.0508	2.6875
0.47	2.0119	1.7335	4.6944	4.1185	2.3333	2.0471	2.7080
0.48	2.0000	1.7178	4.6878	4.0868	2.3439	2.0434	2.7290
0.49	1.9877	1.7019	4.6808	4.0547	2.3549	2.0399	2.7503
0.5	1.9751	1.6858	4.6735	4.0220	2.3662	2.0364	2.7723

References

[1] R. Szilard, *Theory and Analysis of Plates*, Prentice-Hall Inc, Englewood Cliffs, NJ, 1974.
[2] A.W. Leissa, *Vibration of Plates (NASA SP-160)*, US Government Printing Office, Washington, DC, 1969.

[3] ASTM Standard E1875-00e1. Standard Test Method for Dynamic Young's Modulus, Shear Modulus, and Poisson's Ratio by Sonic Resonance, *Book of Standards* Vol. 03.01.
[4] ASTM Standard E1876-01. Standard Test Method for Dynamic Young's Modulus, Shear Modulus, and Poisson's Ratio by Impulse Excitation of Vibration, *Book of Standards* Vol. 03.01.
[5] F.J. Nieves, F. Gascòn, A. Bayòn, Measurement of the dynamic elastic constants of short isotropic cylinder, *Journal of Sound and Vibration* 265 (2003) 917–933.
[6] M. Alfano, L. Pagnotta, An inverse procedure for determining the material constants of isotropic square plates by impulse excitation of vibration, in: M.P. Cartmell, M. Lucas (Eds.), *Advances in Experimental Mechanics IV*, Trans Tech Publication Ltd, 2005, pp. 287–292.
[7] W.P. De Wilde, B. Narmon, H. Sol, M. Roovers, Determination of the material constants of an anisotropic lamina by free vibration analysis, in: *Proceedings of the Second International Modal Analysis Conference*, Orlando, FL, 1984, pp. 44–49.
[8] W.P. De Wilde, H. Sol, M. Van Overmeire, Coupling of Lagrange interpolation modal analysis and sensitivity analysis in the determination of anisotropic plate rigidities, in: *Proceedings of the Fourth International Modal Analysis Conference*, Los Angeles, CA, 1986, pp. 1058–1063.
[9] H. Sol, Identification of Anisotropic Plate Rigidities Using Free Vibration Data, Ph.D. Thesis, University of Brussels, 1986.
[10] L.R. Deobald, R.F. Gibson, Determination of elastic constants of orthotropic plates by modal analysis/Rayleigh–Ritz technique, *Journal of Sound and Vibration* 124 (1988) 269–283.
[11] P. Pedersen, P.S. Frederiksen, Identification of orthotropic material moduli by a combined experimental/numerical method, *Measurement* 10 (1992) 113–118.
[12] T.C. Lai, T.C. Lau, Determination of elastic constants of a generally orthotropic plate by modal analysis, *International Journal of Analytical and Experimental Modal Analysis* 8 (1993) 15–33.
[13] M.E. McIntyre, J. Woodhouse, On measuring the elastic and damping constants of orthotropic sheet materials, *Acta Metallurgica* 36 (1988) 1397–1416.
[14] E.O. Ayorinde, R.F. Gibson, Elastic constants of orthotropic composites materials using plates resonance frequencies, classical lamination theory and an optimized three mode Rayleigh formulation, *Composites Engineering* 3 (1993) 395–407.
[15] K.E. Fällström, M.A. Jonsson, A nondestructive method to determine material properties in anisotropic plates, *Polymer Composites* 12 (1991) 293–305.
[16] K.E. Fällström, M.A. Jonsson, Determining material properties in anisotropic plates using Rayleigh's method, *Polymer Composites* 8 (1991) 306–314.
[17] S.F. Hwang, C.S. Chang, Determination of elastic constants of materials by vibration testing, *Composite Structures* 49 (2000) 183–190.
[18] P. Pedersen, Optimisation method applied to identification of materials parameters, in: H.A. Eschenauer, G. Thierau (Eds.), *Discretization Methods and Structural Optimisation—Procedures and Applications*, Springer, Berlin, 1989, pp. 277–283.
[19] C.M. Mota Soares, M. Moreira de Freitas, A.L. Araùjo, P. Pedersen, Identification of material properties of composite plate specimens, *Composite Structures* 25 (1993) 277–285.
[20] R. Rikards, A. Chate, W. Steinchen, A. Kessler, A.K. Bledzki, Method for identification of elastic properties of laminates based on experiment design, *Composites Part B* 30 (1999) 279–289.
[21] C. Maletta, L. Pagnotta, Determining material properties in anisotropic plates using genetic algorithms and vibration test data, *International Journal of Mechanics and Materials in Design* 1 (2004) 199–211.
[22] G.L. Qian, S.V. Hoa, X. Xiao, A vibration method for measuring mechanical properties of composite, theory and experiment, *Composite Structures* 39 (1997) 31–38.
[23] G.W. Caldersmith, Vibration of orthotropic rectangular plates, *Acustica* 56 (1984) 144–152.
[24] L. Gaul, K. Willner, S. Hurlebaus, Determination of material properties of plates from modal ESPI measurement, in: *Proceedings of the 17th International Modal Analysis Conference*, Orlando, FL, 1999, pp. 1756–1763.
[25] M. Grediac, P.A. Paris, Direct identification of elastic constants of anisotropic plates by modal analysis: theoretical and numerical aspects, *Journal of Sound and Vibration* 195 (1996) 401–415.
[26] M. Grediac, N. Fourier, P.A. Paris, Y. Surrel, Direct identification of elastic constants of anisotropic plates by modal analysis: experiment and results, *Journal of Sound and Vibration* 210 (1996) 645–659.
[27] D. Young, Vibration of rectangular plates by the Ritz method, *Journal of Applied Mechanics* 17 (1950) 448–453.
[28] G.B. Warburton, The vibration of rectangular plates, *Proceedings of the Institution of Mechanical Engineers* 168 (1953) 371–384.
[29] A.W. Leissa, The free vibration of rectangular plates, *Journal of Sound and Vibration* 31 (1973) 257–293.
[30] *MSC/NASTRAN Basic Dynamic Analysis User's Guide*, The MacNeal-Schwendler Corporation, USA, 1997.